

Reconstruction procedures for two inverse scattering problems without the phase information

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Abstract

This is a continuation of two recent publications of the authors [17, 18] about reconstruction procedures for 3-d phaseless inverse scattering problems. The main novelty of this paper is that, unlike [18], the Born approximation for the case of the wave-like equation is not considered. It is shown here that the phaseless inverse scattering problem for the 3-d wave-like equation in the frequency domain leads to the well known Inverse Kinematic Problem. Uniqueness theorem follows. Still, since the Inverse Kinematic Problem is very hard to solve, a linearization is applied. More precisely, geodesic lines are replaced with straight lines. As a result, an approximate explicit reconstruction formula is obtained via the inverse Radon transform. The second reconstruction method is via solving a problem of the integral geometry using integral equations of the Abel type.

Keywords: phaseless inverse scattering, wave equation, reconstruction formula, Radon transform

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1 Introduction

The Phaseless Inverse Scattering Problems (PISPs) arise in applications to imaging of microstructures of sizes of the micron range of less ($1 \text{ micron} = 1\mu m = 10^{-6}m$). In particular, this includes imaging of nano structures of sizes of hundreds of nanometers ($\approx 10^{-7}m$) and biological cells whose sizes are in the range of $(5, 100)\mu m$ [29, 30]. To image these objects, one needs to use either optical radiation with the wavelength less than $1\mu m$ or the X-ray radiation. However, it is well known that only the intensity (i.e. the square modulus) of the corresponding complex valued wave field can be measured for such small wavelengths. The phase cannot be measured, see, e.g. [5, 6, 11, 28, 35]. Therefore, we arrive at the problem of the reconstruction of the scatterer using only the intensity of the

scattered wave field. In this case the propagation of the wave field is governed by the wave-like PDE in the frequency domain.

A similar problem, although for the Schrödinger equation in the frequency domain, arises in the quantum inverse scattering, where only the differential scattering cross-section can be measured, which is actually the square modulus of the solution of that equation, see, e.g. page 8 of [23] and Chapter 10 of [3]. Unlike the wave-like equation (2.6), in the case of the Schrödinger equation the function $\beta(x)$ (see (2.6)) is not multiplied by k^2 , which simplifies the problem. Note that, unlike PISPs, the conventional inverse scattering theory is based on the assumption that both the intensity and the phase of the complex valued wave field are measured, see, e.g. [3, 8, 23, 24].

A reconstruction procedure for a 3-d PISP for the wave-like PDE in the frequency domain was proposed by the authors in [18]. In [18] the linearization based on the Born approximation was used. However, the Born approximation assumption is inconvenient, since it is actually assumed there that $k^2 |\beta(x)| \ll 1$, where $\beta(x)$ is the unknown coefficient and k is the frequency. Hence, the Born approximation breaks down for $k \gg 1$. On the other hand, large values of k are used in the reconstruction formula of [18]. Thus, the goal of this paper is to lift the assumption about the Born approximation. We still linearize the problem. However, the *main novelty* here is that our linearization does not break down when $k \rightarrow \infty$. We achieve this via an extensive use of the structure of the fundamental solution of an associated hyperbolic equation with a variable coefficient in its principal part. The latter is the main new element here as compared with [17, 18]. This structure was derived in [33], also see [32].

The first uniqueness result for the PISP was proven in [13] for the 1-d case with some follow up publications [1, 22]. Uniqueness for the 3-d case was proven in [14, 15, 16]. However, proofs in these references are not constructive. In fact, besides just uniqueness only it is desirable to develop rigorous numerical methods for the phase reconstruction. Many heuristic attempts were made by physicists to reconstruct the phase, see, e.g. [5, 6, 11, 28, 35].

However, rigorous numerical methods for PISPs were derived only very recently by the authors and Novikov [17, 18, 26, 27]. In [17] a 3-d PISP for the Schrödinger equation was considered and an explicit reconstruction formula was derived, which is based on the inverse Radon transform, see, e.g. [21] for this transform. Thus, a long standing problem posed in 1977 in Chapter 10 of [3] was addressed in [17] for the first time. In [18] the result of [17] was extended to the case of the wave-like equation (2.6) using the Born approximation assumption.

While only unknown scatterers are involved in measurements in [17, 18], the reconstruction formula of [26] requires the involvement of two more known scatterers. A quite general reconstruction procedure of [27] is using measurements of the intensity of the full wave field on at least two spheres in the far field zone. The latter is unlike [17, 18, 26], where the intensity of only the scattered wave field is measured on just one surface and the far field approximation is not used. Note that usually the intensity of the scattered rather than the full wave field is measured. This is because measurements are conducted only outside of the brightening area since detectors are “burned” in the brightening area. For example, in images on page 22 of [6] the areas with brightening are depicted in the red color. On the other hand, outside of the brightening area the intensity of the full wave field is well approximated by the intensity of the scattered wave field. While works [14, 15, 16, 17, 18, 26, 27] are concerned with the reconstructions of coefficients of PDEs

from the phaseless scattered data, publications [9, 10] consider numerical methods of the reconstruction of shapes of obstacles from the phaseless scattered data.

We present here two reconstruction methods for two PISPs. In both cases we use measurements of the intensity on a single sphere on many frequencies and the far field approximation is not used. First, we reduce our PISP to the Inverse Kinematic Problem [20, 31, 32]. This leads to a new uniqueness result, which is based on two items: (1) a uniqueness theorem of [32] for the Inverse Kinematic Problem and (2) our reconstruction procedure. Next, we linearize the Inverse Kinematic Problem, as in [20, 31, 32], and reconstruct the unknown coefficient via the inverse Radon transform, as in [17, 18]. In the second approach we obtain after that linearization a problem of the integral geometry and solve it explicitly via solving certain integral equations of the Abel type. In our linearization we assume, similarly with [20, 31, 32], that certain integrals over geodesic lines are actually integrals over straight lines.

As it is often happen to other reconstruction procedures (see, e.g. [26, 27]), our two reconstruction procedures cannot be considered as algorithms yet. In other words, they cannot be considered as sequences of steps leading to the numerical solutions. To turn them in algorithms, our reconstruction steps must be complemented by some regularization steps. Although the latter is possible, we leave this to future numerical publications for brevity.

In section 2 statements of problems under consideration are presented. In section 3 we consider an auxiliary Cauchy problem for a hyperbolic equation, which is important for our study. In section 4 we present a reconstruction method via the inversion of the Radon transform. Finally in section 5 we present the second reconstruction method via solution of a problem of the integral geometry.

2 Problem statement

Let $B > 0$ be a number. Let $\Omega = \{|x| < R\} \subset \mathbb{R}^3$ be the ball of the radius $R < B$ with the center at $\{0\}$. Denote $Y = \{|x| < B\}$, $S = \{|x| = B\}$. Let $n(x)$, $x \in \mathbb{R}^3$ be a real valued function satisfying the following conditions

$$n \in C^{15}(\mathbb{R}^3), \quad n^2(x) = 1 + \beta(x), \quad (2.1)$$

$$\beta(x) \geq 0, \quad \beta(x) = 0 \quad \text{for } x \in \mathbb{R}^3 \setminus \Omega. \quad (2.2)$$

The smoothness requirement imposed on the function $n(x)$ is clarified in the proof of Theorem 1 in subsection 3.1. The conformal Riemannian metric generated by $n(x)$ is

$$d\tau = n(x) |dx|, \quad |dx| = \sqrt{(dx_1)^2 + (dx_2)^2 + (dx_3)^2}. \quad (2.3)$$

Below we impose the following Assumption:

Assumption. We assume that geodesic lines of the metric (2.3) satisfy the regularity condition, i.e. for each two points $x, y \in \mathbb{R}^3$ there exists a single geodesic line $\Gamma(x, y)$ connecting these points.

It is well known from the Hadamard-Cartan theorem [2] that in any simply connected complete manifold with a non positive curvature each two points can be connected by a single geodesic line. The manifold (Ω, n) is called the manifold of a non positive curvature,

if the section curvatures $K(x, \sigma) \leq 0$ for all $x \in \Omega$ and for all two-dimensional planes σ . A sufficient condition for $K(x, \sigma) \leq 0$ was derived in [34]

$$\sum_{i,j=1}^3 \frac{\partial^2 \ln n(x)}{\partial x_i \partial x_j} \xi_i \xi_j \geq 0, \quad \forall x, \xi \in \mathbb{R}^3.$$

For $x, y \in \mathbb{R}^3$, let $\tau(x, y)$ be the solution to the problem

$$|\nabla_x \tau(x, y)|^2 = n^2(x), \quad \tau(x, y) = O(|x - y|), \quad \text{as } y \rightarrow x. \quad (2.4)$$

Let $d\sigma$ be the euclidean arc length of the geodesic line $\Gamma(x, y)$. Then

$$\tau(x, y) = \int_{\Gamma(x, y)} n(\xi) d\sigma. \quad (2.5)$$

Hence, $\tau(x, y)$ is the travel time between points x and y due to the Riemannian metric (2.3). Due the Assumption, $\tau(x, y)$ is a single-valued function of both points x and y in $\mathbb{R}^3 \times \mathbb{R}^3$.

We consider the following equation

$$\Delta u + k^2 n^2(x) u = -\delta(x - y), \quad x \in \mathbb{R}^3, \quad (2.6)$$

where the Laplace operator is taken with respect to x , the frequency $k > 0$ is a positive real number and $y \in \mathbb{R}^3$ is the source position. Naturally, we assume that the function $u(x, k, y)$ satisfies the radiation condition

$$\frac{\partial u}{\partial r} + iku = o(r^{-1}) \quad \text{as } r = |x - y| \rightarrow \infty. \quad (2.7)$$

Denote $u_0(x, k, y)$ the solution of the problem (2.6), (2.7) for the case $n(x) \equiv 1$. Then u_0 is the incident spherical wave,

$$u_0(x, k, y) = \frac{\exp(-ik|x - y|)}{4\pi|x - y|}.$$

Let $u_{sc}(x, k, y)$ be the scattered wave, which is due to the presence of scatterers, in which $n(x) \neq 1$. Then

$$u_{sc}(x, k, y) = u(x, k, y) - u_0(x, k, y) = u(x, k, y) - \frac{\exp(-ik|x - y|)}{4\pi|x - y|}. \quad (2.8)$$

Combining Theorem 8.7 of [4] with Theorem 6.17 of [7] and taking into account that $n \in C^{15}(\mathbb{R}^3)$, we obtain that the problem (2.6), (2.7) has unique solution $u \in C^{16+\alpha}(|x - x_0| \geq \eta)$, $\forall \eta > 0$ for any $\alpha \in (0, 1)$. Here $C^{k+\alpha}$ are Hölder spaces.

We consider the following two Phaseless Inverse Scattering Problems (PISPs):

PISP1. Suppose that the following function $f_1(x, k, y)$ is known

$$f_1(x, k, y) = |u_{sc}(x, k, y)|^2, \quad \forall (x, y) \in S \times S, \quad \forall k \geq k_0, \quad (2.9)$$

where $k_0 = \text{const.} > 0$. Determine the function $\beta(x)$ satisfying conditions (2.1), (2.2).

PISP2. Suppose that the following function $f_2(x, k, y)$ is known

$$f_2(x, k, y) = |u(x, k, y)|^2, \quad \forall (x, y) \in S \times S, \quad \forall k \geq k_0, \quad (2.10)$$

where $k_0 = \text{const.} > 0$. Determine the function $\beta(x)$ satisfying conditions (2.1), (2.2).

In both these problems we do not take into account the fact that the intensity of the wave field can be measured only outside of the brightening area (see Introduction). This case is a more difficult one than we now consider and we hope to study it in the future.

Below we present a reconstruction method for the PISP1 in a linearized case. This method is based on the inverse Radon transform. In addition, we present a different reconstruction procedure both for the PISP1 and for the PISP2 (both are linearized) via solving a problem of the integral geometry. Note that while the inverse Radon transform is applicable to the PISP1, it is inapplicable to the PISP2. On the other hand, the reconstruction methods based on the integral geometry work for both these problems.

Remark. In fact, it follows from our reconstruction procedures that, in the linearized case, it is sufficient to know functions $f_1(x, k, y)$, $f_2(x, k, y)$ only for all $(x, y) \in (S \cap \{x_3 = z\}) \times (S \cap \{x_3 = z\})$ for each $z \in (-R, R)$ and for all $k \geq k_0$. This case of data collection is more economical than the one of (2.9), (2.10). In this case the function $\beta(x)$ is reconstructed separately in each 2-d cross-section $\Omega \cap \{x_3 = z\}$ of the domain Ω .

3 The auxiliary Cauchy problem for a hyperbolic equation

Consider the following Cauchy problem

$$n^2(x)v_{tt} = \Delta v + \delta(x - y, t), \quad x \in \mathbb{R}^3, t > 0, \quad (3.1)$$

$$v(x, 0) = v_t(x, 0) = 0. \quad (3.2)$$

3.1 The form of the solution of the problem (3.1), (3.2)

Let $\zeta = (\zeta_1, \zeta_2, \zeta_3)$, $\zeta = \zeta(x, y)$ be geodesic coordinates of a variable point x with respect to a fixed point y in the above Riemannian metric (2.3). By the above Assumption, there exists a one-to-one correspondence $x \Leftrightarrow \zeta$ for any fixed y . Therefore, for any fixed point y the function $\zeta = \zeta(x, y)$ has the inverse function $x = f(\zeta, y)$ which determines the geodesic line $\Gamma(x, y) = \{\xi : \xi = f(s\zeta_0, y), s \in [0, \tau(x, y)]\}$, where ζ_0 is the vector which is tangent to $\Gamma(x, y)$ at the point y , is directed towards the point x and also $|\zeta_0| = n^{-1}(y)$. Moreover, by the formula (2.2.28) in the book [33] the function $\zeta(x, y)$ can be expressed via the function $\tau(x, y)$ as

$$\zeta(x, y) = -\frac{1}{2n^2(y)} \nabla_y \tau^2(x, y). \quad (3.3)$$

Note that in our case one should take in the formula (2.2.28) of [33] $A(y) = n^{-2}(y)I$, where I is the unit matrix. Also, since by (2.1) $n(x) \in C^{15}(\mathbb{R}^3)$ and the Assumption holds, then $\tau^2(x, y) \in C^{15}(\mathbb{R}^3 \times \mathbb{R}^3)$, $k \geq 2$, and $\zeta(x, y) \in C^{14}(\mathbb{R}^3 \times \mathbb{R}^3)$ [33]. Consider the Jacobian $J(x, y)$,

$$J(x, y) = \det \frac{\partial \zeta}{\partial x}. \quad (3.4)$$

By the formula (2.2.18) in [33] $\zeta = x - y + O(|x - y|^2)$ as $x \rightarrow y$. Hence,

$$\left. \frac{\partial \zeta_i}{\partial x_j} \right|_{x=y} = \lim_{x_j - y_j = h \rightarrow 0} \frac{h \delta_{ij} + O(h^2)}{h} = \delta_{ij}, \quad i, j = 1, 2, 3,$$

where δ_{ij} is the Kronecker delta. Hence, $J(y, y) = 1$. Next, the smoothness of the function $\zeta(x, y)$ with respect to x implies that the function $J(x, y)$ is continuous with respect to x . Further, since the correspondence $x \Leftrightarrow \zeta$ is one-to-one and differentiable, then (3.4) implies that $J(x, y) \neq 0, \forall (x, y) \in \mathbb{R}^3 \times \mathbb{R}^3$. Hence,

$$J(x, y) > 0, \forall (x, y) \in \mathbb{R}^3 \times \mathbb{R}^3. \quad (3.5)$$

For any $y \in \mathbb{R}^3$ and for any $T > 0$ consider two domains $D(y, T)$ and $D^*(y, T)$,

$$D(y, T) = \{(x, t) : 0 < t \leq T - |x - y|\} \text{ and } D^*(y, T) = \{(x, t) : \tau(x, y) \leq t \leq T - \tau(x, y)\}.$$

Theorem 1. *Let the function $n(x)$ satisfies conditions (2.1) and (2.2) and let the Assumption holds. Then for any $T > 0$ and for any fixed $y \in \mathbb{R}^3$ there exists unique solution $v(x, t, y)$ to the problem (3.1), (3.2), which can be represented in the domain $D(y, T)$ in the form*

$$v(x, t, y) = A(x, y)\delta(t - \tau(x, y)) + \hat{v}(x, t, y)H(t - \tau(x, y)), \quad (3.6)$$

where $A(x, y) > 0$ is given by the formula

$$A(x, y) = \frac{n^2(y)\sqrt{J(x, y)}}{4\pi n(x)\tau(x, y)}, \quad (3.7)$$

$H(t)$ is the Heaviside function and the function $\hat{v}(x, t, y) \in C^2(D^*(y, T))$.

Proof. We note first that we can use $\sqrt{J(x, y)}$ in (3.7) since by (3.5) $J(x, y) > 0$. Rewrite the problem (3.1), (3.2) in the form

$$v_{tt} - \operatorname{div}(n^{-2}(x)\nabla v) + \nabla n^{-2}(x) \cdot \nabla v = n^{-2}(y)\delta(x - y, t), \quad v|_{t < 0} \equiv 0, \quad (3.8)$$

We now use results obtained in the book [33]. Represent the solution to the problem (3.8) for $t \geq 0$ in the form

$$v(x, t, y) = \frac{1}{n^2(y)} \left[a_{-1}(x, y)\delta(t^2 - \tau^2(x, y)) + \sum_{k=0}^s a_k(x, y)\theta_k(t^2 - \tau^2(x, y)) + w_s(x, t, y) \right], \quad (3.9)$$

where the integer $s > 1$ is chosen below and functions $\theta_k(t)$ are defined as

$$\theta_k(t) = \frac{t^k}{k!}H(t), \quad k = 0, 1, 2, \dots, s.$$

Coefficients $a_k(x, y)$ in (3.9) are defined by the following formulas

$$a_{-1}(x, y) = \frac{\sqrt{J(x, y)}n^4(y)}{2\pi n(x)}, \quad (3.10)$$

$$a_r(x, y) = \frac{a_{-1}(x, y)}{4\tau^{r-1}(x, y)} \int_{\Gamma(x, y)} \tau^r(\xi, y)(a_{-1}(\xi, y))^{-1}n^{-1}(\xi)\Delta_\xi a_{r-1}(\xi, y)d\sigma, \quad r \geq 0, \quad (3.11)$$

where ξ is a variable point on this geodesic line $\Gamma(x, y)$. We have derived formulae (3.10), (3.11) from formulas (2.2.41)-(2.2.44) of the book [33], in which the notation α_r was used for a_r and $d\tau' = n(\xi)d\sigma$. We took $m = 1$ in the latter formulae.

The residual function $w_s(x, t, y)$ in (3.9) is the solution of the following problem

$$\partial^2 w_s - n^{-2}(x)\Delta w_s = F_s(x, t, y), \quad w|_{t < 0} \equiv 0, \quad (3.12)$$

$$F_s(x, t, y) = n^{-2}(x)(\Delta a_s(x, y))\theta_s(t^2 - \tau^2(x, y)). \quad (3.13)$$

Let $k > 1$ be a sufficiently large integer which is chosen below. If the function $n(x) \in C^k(\mathbb{R}^3)$, then $\tau^2(x, y) \in C^k(\mathbb{R}^3 \times \mathbb{R}^3)$ (see [33], p. 27). Therefore it follows from (3.10) and (3.11) that

$$a_{-1}(x, y) \in C^{k-2}(\mathbb{R}^3 \times \mathbb{R}^3), \quad a_r(x, y) \in C^{k-4-2r}(\mathbb{R}^3 \times \mathbb{R}^3), \quad r = 0, 1, \dots, s.$$

Hence, the function $F_s(x, t, y) \in C^l(D(y, T))$, where $l = \min(m - 6 - 2s, s)$ and this function vanishes for $t \leq \tau(x, y)$. Hence, $w_s(x, t, y) \equiv 0$ for $t \leq \tau(x, y)$.

Using theorems 3.2, 4.1, Corollary 4.2 and energy estimates of Chapter 4 of [19], one can easily prove that $w_s \in C^{l-1}(D(y, T))$, see, e.g. Theorem 2.2 of [12] for a similar result. Choosing $m = 15$ and $s = 3$, we obtain $l = 3$. Thus, $w_3 \in C^2(D(y, T))$. Since by (3.7) and (3.10) $A(x, y) = a_{-1}(x, y)/(2n^2(y)\tau(x, y))$, then $A(x, y) > 0$ and $A(x, y) \in C^{13}(\mathbb{R}^3 \times \mathbb{R}^3)$. Moreover, setting

$$\hat{v}(x, t, y) = \frac{1}{n^2(y)} \left[\sum_{r=0}^3 a_r(x, y) \frac{(t^2 - \tau^2(x, y))^r}{r!} + w_3(x, t, y) \right], \quad (x, y) \in D^*(T, y),$$

we see that formula (3.9) coincides with formula (3.6) and also that the function $\hat{v}(x, t, y) \in C^2(D^*(T, y))$. \square

3.2 Connection with the problem (2.6), (2.7)

Let $\Phi \subset \mathbb{R}^3$ be an arbitrary bounded domain. Lemma 6 of Chapter 10 of the book [37] as well as Remark 3 after that lemma guarantee that functions $\partial_t^k v(x, t, \nu)$, $k = 0, 1, 2$ and $\Delta_x v(x, t, \nu)$ decay exponentially as $t \rightarrow \infty$, as long as x remains in the domain K . In other words, there exist constants $M = M(\Phi, \beta) > 0$, $c = c(\Phi, \beta) > 0$ such that

$$|\partial_t^k v(x, t, y)|, |\Delta v(x, t, y)| \leq M e^{-ct} \text{ for all } t \geq t_0 \text{ and for all } x \in \Phi, \quad (3.14)$$

where $t_0 = t_0(\Phi, \beta) = \text{const.} > 0$. Hence, one can consider the Fourier transform $V(x, k, y)$ of the function v ,

$$V(x, k, y) = \int_0^\infty v(x, t, y) \exp(-ikt) dt. \quad (3.15)$$

Next, Theorem 3.3 of [36] and Theorem 6 of Chapter 9 of [37] guarantee that $V(x, k, y) = u(x, k, y)$, where the function $u(x, k, y)$ is the above solution of the problem (2.6), (2.7).

Comparing (3.14) and (3.15) with (3.6), integrating by parts in (3.15) and also using (2.8), we obtain

$$u(x, k, y) = \exp(-ik\tau(x, y)) \left[A(x, y) + O\left(\frac{1}{k}\right) \right], \quad k \rightarrow \infty, \quad (3.16)$$

$$u_{sc}(x, k, y) = A(x, y) \exp(-ik\tau(x, y)) - \frac{\exp(-ik|x-y|)}{4\pi|x-y|} + O\left(\frac{1}{k}\right), k \rightarrow \infty. \quad (3.17)$$

Hence, (2.9), (2.10), (3.16) and (3.17) imply that for $(x, y) \in S \times S$

$$f_1(x, k, y) = A^2(x, y) + \frac{1}{16\pi^2|x-y|^2} - \frac{A(x, y)}{2\pi|x-y|} \cos[k(\tau(x, y) - |x-y|)] + O\left(\frac{1}{k}\right), k \rightarrow \infty, \quad (3.18)$$

$$f_2(x, k, y) = A^2(x, y) + O\left(\frac{1}{k}\right), k \rightarrow \infty. \quad (3.19)$$

4 Approximate solution of the linearized PISP1 via the inverse Radon transform

4.1 Reconstruction of the function $\tau(x, y)$ for $(x, y) \in S \times S$

Below in this subsection $k \geq k_1 > k_0$, where $k_1 \gg 1$ is a sufficiently large number. We now fix the point $(x, y) \in S \times S$ and consider $f_1(x, k, y)$ as the function of k for $k \geq k_1$. It is possible to figure out whether or not $\tau(x, y) = |x-y|$. Indeed, it follows from (3.19) that $\tau(x, y) = |x-y|$ if and only if $\lim_{k \rightarrow \infty} f_1(x, k, y)$ exists. Ignoring in (3.18) the term $O(k^{-1})$, we obtain for $k \geq k_1$

$$f_1(x, k, y) = A^2(x, y) + \frac{1}{16\pi^2|x-y|^2} - \frac{A(x, y)}{2\pi|x-y|} \cos[k(\tau(x, y) - |x-y|)]. \quad (4.1)$$

In (4.1) we use “=” instead of “ \approx ”. By (4.1) there exists a number $k_2 \geq k_1$ such that

$$f_1^*(x, y) = f_1(x, k_2, y) = \max_{k \geq k_1} f_1(x, k, y) = \left(A(x, y) + \frac{1}{4\pi|x-y|} \right)^2 \quad (4.2)$$

Hence, we find the number $A(x, y)$ as

$$A(x, y) = \sqrt{f_1^*(x, y)} - \frac{1}{4\pi|x-y|}. \quad (4.3)$$

Assume that $\tau(x, y) \neq |x-y|$. Then the positivity of the function $\beta(x)$ and (2.5) imply that $\tau(x, y) > |x-y|$. Choose the number $k_3 > k_2$ such that

$$k_3 = \min \{k : k > k_2, f_1(x, k, y) = f_1^*(x, y)\}. \quad (4.4)$$

Then (4.1)-(4.4) imply that

$$k_3(\tau(x, y) - |x-y|) = k_2(\tau(x, y) - |x-y|) + 2\pi.$$

Therefore, we reconstruct the number $\tau(x, y)$ as

$$\tau(x, y) = |x-y| + \frac{2\pi}{k_3 - k_2}. \quad (4.5)$$

Next, we should reconstruct the function $\beta(x)$, which is done below. To do this, we consider a linearization. However, we now can formulate uniqueness theorem for PISP1 without the linearization assumption. This is because the knowledge of the function $\tau(x, y)$ for all $(x, y) \in S \times S$, which follows from (4.5), is equivalent to the so-called Inverse Kinematic Problem. This problem was studied in [20, 31, 32]. In particular, Theorem 3.4 of Chapter 3 of [32] claims uniqueness of the reconstruction of the function $n(x)$ from the knowledge of the function $\tau(x, y)$ for all $(x, y) \in S \times S$. Thus, our Theorem 2 follows immediately from Theorem 3.4 of Chapter 3 of [32] as well as from (4.5).

Theorem 2 (uniqueness). *Suppose that two functions $n_1(x), n_2(x)$ satisfying conditions (2.1), (2.2) also satisfy Assumption of section 2. In addition, assume that these two functions generate the same function $f_1(x, k, y)$ in (2.9) and that the function $f_1(x, k, y)$ has the form (4.1), i.e. the term $O(1/k)$ in (3.18) is dropped. Then $n_1(x) \equiv n_2(x)$.*

4.2 Reconstruction of the function $\beta(x)$

Assume that

$$\|\beta\|_{C^2(\overline{\Omega})} \ll 1. \quad (4.6)$$

Then the linearization of the function $\tau(x, y)$ with respect to the function β leads to

$$\tau(x, y) = |x - y| + \int_{L(x, y)} \beta(\xi) d\sigma, \quad (4.7)$$

where $L(x, y)$ is the segment of the straight line connecting points x and y and $d\sigma$ is its arc length, see Theorem 11 in Chapter 3 of [20] as well as §5 in Chapter 2 of [31] and §4 in Chapter 3 of [32]. To be precise, in (4.7) one should have “ \approx ” instead of “ $=$ ”. Since the function $\tau(x, y)$ was approximately reconstructed via (4.5), then we obtain from (4.7) that the following function $h(x, y) = \tau(x, y) - |x - y|$ is known

$$h(x, y) = \int_{L(x, y)} \beta(\xi) ds, \quad \forall (x, y) \in S \times S. \quad (4.8)$$

For any number $z \in \mathbb{R}$ consider the plane $P_z = \{x_3 = z\}$. Consider the disk $Q_z = \overline{Y} \cap P_z$ and let $S_z = S \cap P_z$ be its boundary. Clearly $Q_z \neq \emptyset$ for $z \in (-B, B)$ and $Q_z = \emptyset$ for $|z| \geq B$. Denote $0_z = (0, 0, z) \in Q_z$ the orthogonal projection of the origin on the plane P_z . For an arbitrary $z \in (-B, B)$ denote $B_z = \sqrt{B^2 - z^2}$ the radius of the circle S_z . We have

$$Y = \bigcup_{z=-B}^B Q_z, \partial Y := S = \bigcup_{z=-B}^B S_z, \Omega \subset Y.$$

We now introduce some notations of the Radon transform, which we take from the book [21]. Since our reconstruction formula is based on the inversion of the two-dimensional Radon transform, we now parametrize $L(x, y)$ in the conventional way of the parametrization of the Radon transform [21]. Let ν be the unit normal vector to $L(x, y)$ lying in the plane P_z and pointing outside of the point 0_z . Let $\alpha \in (0, 2\pi]$ be the angle between ν and the x_1 -axis. Then $\nu = \nu(\alpha) = (\cos \alpha, \sin \alpha)$ (it is convenient here to discount the third coordinate of ν , which is zero). Let s be the signed distance between $L(x, y)$ and the

point 0_z (page 9 of [21]). It is clear that there exists a one-to-one correspondence between pairs (x, y) and $(\nu(\alpha), s)$,

$$(x, y) \Leftrightarrow (\nu(\alpha), s), (x, y) \in S_z \times S_z, \alpha = \alpha(x, y) \in (0, 2\pi], s = s(x, y) \in (-B_z, B_z). \quad (4.9)$$

Hence, we can write

$$L(x, y) = \{r_a = (r_1, r_2, a) : \langle r, \nu(\alpha) \rangle = s\}, \quad (4.10)$$

where $r = (r_1, r_2) \in \mathbb{R}^2$, $\langle \cdot, \cdot \rangle$ is the scalar product in \mathbb{R}^2 and parameters $\alpha = \alpha(x, y)$ and $s = s(x, y)$ are defined as in (4.9).

Consider an arbitrary function $g = g(r) \in C^4(P_z)$ such that $g(r) = 0$ for $y \in P_z \setminus Q_z$. Hence,

$$\int_{L(x, y)} g(r) d\sigma = \int_{\langle r, \nu(\alpha) \rangle = s} g(r) d\sigma, \forall (x, y) \in S_z \times S_z, \quad (4.11)$$

where $\alpha = \alpha(x, y)$, $s = s(x, y)$ are as in (4.9). The parametrization of $L(x, y)$ in (4.11) is as in (4.10). Therefore, using (4.9)-(4.11), we can define the 2-d Radon transform Rg of the function g as

$$(Rg)(x, y) = (Rg)(\alpha, s) = \int_{\langle r, \nu(\alpha) \rangle = s} g(r) d\sigma, \forall (x, y) \in S_z \times S_z.$$

Hence, by (4.8) and (4.9)

$$h(x, y) = (R\beta)(\alpha, s) = \int_{\langle r, \nu(\alpha) \rangle = s} \beta(r, a) d\sigma, \forall (x, y) \in S_z \times S_z, \forall z \in (-B, B). \quad (4.12)$$

Finally, since the function $h(x, y)$ is known, then (4.12) leads to the following reconstruction formula

$$\beta(r_1, r_2, a) = R^{-1}(h(x, y))(r_1, r_2, a), (x, y) \in S_z \times S_z, \forall z \in (-B, B). \quad (4.13)$$

Formula (4.13) is our *final reconstruction result* for PISP1 via the inverse Radon transform. Since the formula for R^{-1} is well known [21], we do not specify it here for brevity.

We now formulate uniqueness result for the linearized PISP1. This result follows immediately from (4.12).

Theorem 3 (uniqueness). *Suppose that two functions $\beta_1(x), \beta_2(x)$ satisfying conditions (2.1), (2.2). Also, assume that corresponding functions $n_1(x), n_2(x)$ satisfy Assumption of section 2. In addition, assume that these two functions generate the same function $f_1(x, k, y)$ in (2.9) and that the function $f_1(x, k, y)$ has the form (4.1), i.e. the term $O(1/k)$ in (3.18) is ignored. Finally, assume that the linearization (4.7) is valid. Then $\beta_1(x) \equiv \beta_2(x)$.*

5 Reconstruction via the integral geometry

In this section we present the reconstruction method for the function $\beta(x)$ from the function $A(x, y)$ given for all $(x, y) \in S \times S$. This method works for both PISP1 and PISP2. Indeed, consider two cases:

Case 1: PISP1. Given the function $f_1(x, k, y)$ in (2.9) and dropping the term $O(1/k)$ in (3.18), we obtain the function $A(x, y)$ via (4.3).

Case 1: PISP1. Recall that by (2.1) and (2.2) $n(x) = 1$ for $x \in S$. Hence, in this case (2.10), (3.7) and (3.19) imply that

$$\lim_{k \rightarrow \infty} \sqrt{f_2(x, k, y)} = A(x, y) = \frac{\sqrt{J(x, y)}}{4\pi\tau(x, y)}, \quad \forall (x, y) \in S \times S, \quad (5.1)$$

where the determinant $J(x, y)$ was defined in (3.4).

5.1 Derivation of the problem of the integral geometry

We again use (4.6) and the linearization (4.7). Rewrite (4.7) as

$$\tau(x, y) = |x - y| \left(1 + \int_0^1 \beta(y + s(x - y)) ds \right). \quad (5.2)$$

Because of (4.6), we ignore in our approximate formulas below all terms containing the square of the integral in (5.2) as well as products of its derivatives. First, using (3.4), we find an approximate formula for the determinant $J(x, y)$ in (5.1). Using (3.3), we obtain the following approximate formula for $\zeta(x, y)$

$$\begin{aligned} \zeta(x, y) = & \frac{(x - y)}{n^2(y)} \left(1 + 2 \int_0^1 \beta(y + s(x - y)) ds \right) \\ & - \frac{|x - y|^2}{n^2(y)} \int_0^1 \nabla \beta(y + s(x - y))(1 - s) ds. \end{aligned} \quad (5.3)$$

By (5.3) we have for $i, j = 1, 2, 3$

$$\begin{aligned} \frac{\partial \zeta_i(x, y)}{\partial x_j} = & \frac{\delta_{ij}}{n^2(y)} \left[1 + 2 \int_0^1 \beta(y + s(x - y)) ds \right] \\ & + \frac{2}{n^2(y)} \left[(x_i - y_i) \int_0^1 \beta_{x_j}(y + s(x - y)) s ds - (x_j - y_j) \int_0^1 \beta_{x_i}(y + s(x - y))(1 - s) ds \right] \\ & - \frac{|x - y|^2}{n^2(y)} \int_0^1 \beta_{x_i x_j}(y + s(x - y)) s(1 - s) ds. \end{aligned}$$

Hence, in the 3×3 determinant $J(x, y) = \det(\partial \zeta / \partial x)$ the product of diagonal terms dominates the rest of terms, which should be set to zero because of products of the above mentioned integrals with the function β and its derivatives. Hence, an approximate formula for $J(x, y)$ is

$$J(x, y) = \frac{1}{n^6(y)} \left[1 + 6 \int_0^1 \beta(y + s(x - y)) ds + 2(x - y) \int_0^1 \nabla \beta(y + s(x - y))(2s - 1) ds \right]$$

$$-|x-y|^2 \int_0^1 \Delta\beta(y+s(x-y))s(1-s)ds \Bigg]. \quad (5.4)$$

Note that

$$(x-y)\nabla\beta(y+s(x-y)) = \frac{\partial\beta(y+s(x-y))}{\partial s}.$$

Hence,

$$(x-y) \int_0^1 \nabla\beta(y+s(x-y))(2s-1)ds = \beta(x) + \beta(y) - 2 \int_0^1 \beta(y+s(x-y))ds.$$

Hence, we obtain

$$\begin{aligned} J(x, y) &= 1 + 2 \int_0^1 \beta(y+s(x-y))ds \\ &\quad - |x-y|^2 \int_0^1 \Delta\beta(y+s(x-y))s(1-s)ds, \quad (x, y) \in S \times S. \end{aligned} \quad (5.5)$$

Next, by (5.2)

$$\frac{1}{\tau(x, y)} = \frac{1}{|x-y|} \left(1 - \int_0^1 \beta(y+s(x-y))ds \right), \quad (5.6)$$

Thus, (5.1), (5.4) and (5.5) imply that

$$A(x, y) = \frac{1}{4\pi|x-y|} \left(1 - \frac{|x-y|^2}{2} \int_0^1 \Delta\beta(y+s(x-y))s(1-s)ds \right), \quad (x, y) \in S \times S. \quad (5.7)$$

Both in (5.6) and (5.7) we again use “=” instead of “ \approx ”. Denote $\Delta\beta(x) = q(x)$. Then we obtain the following problem:

The Problem of the Integral Geometry. *Find the function $q \in C(\overline{Y})$, $q(x) = 0$ for $x \in Y \setminus \Omega$, assuming that the following integrals $g(x, y)$ are given over segments $L(x, y)$ of straight lines connecting any two points $(x, y) \in S \times S$, $x \neq y$*

$$|x-y| \int_0^1 q(y+s(x-y))s(1-s)ds = g(x, y), \quad (5.8)$$

$$g(x, y) = -8\pi A(x, y) + 2|x-y|^{-1}.$$

Note that $g(x, y) = g(y, x)$. Since the weight function $s(1-s)$ is present in the integral (5.8), the problem (5.8) cannot be solved via the inversion of the Radon transform as in section 4.2. Therefore, we propose a different method in subsection 5.2. If the problem (5.8) is solved, then the function $\beta(x)$ can be found via the solution of the Dirichlet boundary value problem for the Laplace equation,

$$\Delta\beta(x) = q(x), \quad x \in Y; \quad \beta(x)|_S = 0.$$

5.2 Solution of the problem of the integral geometry

We now show that the function $q(x)$ can be reconstructed from (5.8) separately in each 2-d cross-section $Q_z = \overline{Y} \cap P_z, z \in (-R, R)$ of the ball Y . First, we rewrite equation (5.8) as

$$\int_0^{|x-y|} q\left(y + s_1 \frac{(x-y)}{|x-y|}\right) s_1 (|x-y| - s_1) ds_1 = g(x, y), \quad (5.9)$$

$$\forall (x, y) \in S_z \times S_z, x \neq y,$$

where s_1 is the arc length of $L(x, y)$. Recall that the plane $P_z = \{x_3 = z\}$. We consider $z \in (-R, R)$, since $q(x) = 0$ for $|x| > R$ (see (2.2)). In the plane P_z we introduce polar coordinates r, φ of the variable point $\xi = (\xi_1, \xi_2)$ as $\xi_1 = r \cos \varphi, \xi_2 = r \sin \varphi$. We characterize the segment of the straight line $L(x, y)$ passing through points $x, y \in S_z$ by the polar coordinates (ρ, α) of its middle point $(x + y)/2$. Hence, $|x - y| = 2\sqrt{B^2 - z^2 - \rho^2}$. Change variables in the integral (5.9) as

$$s_1 \iff r, s_1 = \sqrt{B^2 - z^2 - \rho^2} - \sqrt{r^2 - \rho^2},$$

Then $s_1(|x - y| - s_1) = B^2 - z^2 - r^2$. The equation of $L(x, y)$ can be rewritten as

$$\varphi = \alpha + (-1)^j \arccos \frac{\rho}{r}, \quad j = 1, 2, \quad r \geq \rho, \quad (5.10)$$

where $j = 1$ corresponds to the part of the straight line $L(x, y)$ between points $(x + y)/2$ and x , and $j = 2$ to the rest of $L(x, y)$.

Note that $ds_1 = -rdr/\sqrt{r^2 - \rho^2}$. Obviously there exists a one-to-one correspondence, up to the symmetry mapping $(x, y) \Leftrightarrow (y, x)$ between pairs $(x, y) \in S_z \times S_z$ and $(\rho, \alpha) \in (0, R) \times (0, 2\pi)$. Denote $q(r \cos \varphi, r \sin \varphi, z) = \tilde{q}(r, \varphi, z)$ and $g(x, y) = \tilde{g}(\rho, \alpha, z)$. Using (5.10), we rewrite equation (5.9) as

$$\sum_{j=1}^2 \int_{\rho}^{\rho_0} \tilde{q}(r, \alpha + (-1)^j \arccos \frac{\rho}{r}, z) \frac{(B^2 - z^2 - r^2) r dr}{\sqrt{r^2 - \rho^2}} = \tilde{g}(\rho, \alpha, z), \quad (5.11)$$

where $\rho_0 = \sqrt{B^2 - z^2}$. Represent functions $\tilde{q}(r, \varphi, z)$ and $\tilde{g}(\rho, \alpha, z)$ via Fourier series,

$$\tilde{q}(r, \varphi, z) = \sum_{n=-\infty}^{\infty} \tilde{q}_n(r, z) \exp(in\varphi), \quad (5.12)$$

$$\tilde{g}(\rho, \alpha, z) = \sum_{n=-\infty}^{\infty} \tilde{g}_n(\rho, z) \exp(in\alpha). \quad (5.13)$$

Multiplying both sides of (5.11) by $\exp(-in\alpha)/(2\pi)$ and integrating with respect to α , we obtain for all $n = 0, \pm 1, \pm 2, \dots$

$$\int_{\rho}^{\rho_0} \tilde{q}_n(r, z) \cos\left(n \arccos \frac{\rho}{r}\right) \frac{2(B^2 - z^2 - r^2) r dr}{\sqrt{r^2 - \rho^2}} = \tilde{g}_n(\rho, z). \quad (5.14)$$

Denote

$$p_n(r, z) = \tilde{q}_n(r, z)(B^2 - z^2 - r^2)r.$$

Then equation (5.14) becomes

$$\int_{\rho}^{\rho_0} p_n(r, z) \cos \left(n \arccos \frac{\rho}{r} \right) \frac{2dr}{\sqrt{r^2 - \rho^2}} = \tilde{g}_n(\rho, z), \quad \rho \in (0, \rho_0]. \quad (5.15)$$

This is the integral equation of the Abel type. To solve equation (5.15), we apply first the operator L to both sides of (5.15), where

$$L(h(\rho))(s) = \frac{1}{\pi} \int_s^{\rho_0} \frac{h(\rho)\rho d\rho}{\sqrt{\rho^2 - s^2}}, \quad s \in (0, \rho_0).$$

Then changing the limits of the integration, we obtain

$$\frac{1}{\pi} \int_s^{\rho_0} p_n(r, z) \left[\int_s^r \frac{2\rho}{\sqrt{\rho^2 - s^2} \cdot \sqrt{r^2 - \rho^2}} \cos \left(n \arccos \frac{\rho}{r} \right) d\rho \right] dr = L(\tilde{g}_n(\rho, z))(s). \quad (5.16)$$

Change variables in the inner integral (5.16) as

$$\rho \Leftrightarrow \theta, \rho^2 = s^2 \cos^2(\theta/2) + r^2 \sin^2(\theta/2).$$

Then

$$2\rho d\rho = (r^2 - s^2) \sin \theta \cos \theta d\theta, \\ \sqrt{\rho^2 - s^2} \cdot \sqrt{r^2 - \rho^2} = (r^2 - s^2) \sin \theta \cos \theta.$$

Hence, equation (5.16) can be rewritten as

$$\int_s^{\rho_0} p_n(r, z) Q_n(r, s) dr = L(\tilde{g}_n(\rho, z))(s), \quad (5.17) \\ Q_n(r, s) = \frac{1}{\pi} \int_0^{\pi} \cos \left(n \arccos \frac{\sqrt{r^2 \cos^2(\theta/2) + s^2 \sin^2(\theta/2)}}{r} \right) d\theta.$$

We have $Q_n(s, s) = 1$. Hence, differentiating (5.17) with respect to s , we obtain Volterra integral equation of the second kind

$$p_n(s, z) - \int_s^{\rho_0} p_n(r, z) T_n(r, s) dr = -\frac{\partial}{\partial s} [L(\tilde{g}_n(\rho, z))(s)], \quad s \in (0, \rho_0), \quad (5.18)$$

$$T_n(r, s) = \frac{ns}{\pi \sqrt{r^2 - s^2}} \int_0^{\pi} \sin \left(n \arccos \frac{\sqrt{r^2 \cos^2(\theta/2) + s^2 \sin^2(\theta/2)}}{r} \right) \\ \times \frac{\sin(\theta/2) d\theta}{\sqrt{r^2 \cos^2(\theta/2) + s^2 \sin^2(\theta/2)}}. \quad (5.19)$$

It follows from (5.19) that the kernel of the Volterra integral equation (5.18) has the form

$$T_n(r, s) = \frac{\tilde{T}_n(r, s)}{\sqrt{r^2 - s^2}},$$

where the function $\tilde{T}_n(r, s)$ is continuous for $0 \leq s \leq r \leq \rho_0$. Therefore, it follows from the theory of Volterra integral equations of the second kind that for each $z \in (-R, R)$ there exists a solution $p_n(s, z) \in C[0, \rho_0]$ of equation (5.18) and this solution is unique. Furthermore, it is well known from that theory that equation (5.18) can be solved iteratively as

$$p_n^0(s, z) = -\frac{\partial}{\partial s} [L(\tilde{g}_n(\rho, z))(s)], \quad (5.20)$$

$$p_n^k(s, z) = \int_s^{\rho_0} p_n^{k-1}(r, z) T_n(r, s) dr - \frac{\partial}{\partial s} [L(\tilde{g}_n(\rho, z))(s)], \quad k = 1, 2, \dots \quad (5.21)$$

and this process converges in the space $C[0, \rho_0]$ to the solution $p_n(s, z)$ of equation (5.18). Formulae (5.20) and (5.21) finish our second reconstruction procedure.

As a corollary, we formulate the following uniqueness theorem, which follows immediately from the above reconstruction process.

Theorem 4 (uniqueness). *Assume that the function $A(x, y)$ has its approximate form (5.7). Suppose that this function is given for all $(x, y) \in S \times S$. Then there exists at most one function $\beta \in C^2(\overline{Y})$, $\beta(x) = 0$ in $Y \setminus \Omega$ which is involved in (5.7). In particular, PISP2 has at most one solution as long as the function $A(x, y)$ is as in (5.7). The same is true for PISP1, if, in addition, the term $O(1/k)$ is dropped in (3.18). \square*

References

- [1] T. Aktosun and P.E. Sacks, Inverse problem on the line without phase information, *Inverse Problems*, 14, 211-224, 1998.
- [2] W. Ballmann, *Lecture on Spaces of Nonpositive Curvature* (DMV-Seminar, Band 25), Birkhäuser Verlag, Basel, 1995.
- [3] K. Chadán and P.C. Sabatier, *Inverse Problems in Quantum Scattering Theory*, Springer-Verlag, New York, 1977.
- [4] D. Colton and R. Kress, *Inverse Acoustic and Electromagnetic Scattering Theory*, Springer-Verlag, New York, 1992.
- [5] A.V. Darahanau, A.Y. Nikulin, A. Souvorov, Y. Nishino, B.C. Muddle and T. Ishikawa, Nano-resolution profiling of micro-structures using quantitative X-ray phase retrieval from Fraunhofer diffraction data, *Physics Letters A*, 335, 494-498, 2005.
- [6] M. Dierolf, O. Bank, S. Kynde, P. Thibault, I. Johnson, A. Menzel, K. Jefimovs, C. David, O. Marti and F. Pfeiffer, Ptychography & lenseless X-ray imaging, *Europhysics News*, 39, 22-24, 2008.

- [7] D. Gilbarg and N.S. Trudinger, *Elliptic Partial Differential Equations of Second Order*, Springer, New York, 1984.
- [8] V. Isakov, *Inverse Problems for Partial Differential Equations*, Second Edition, Springer, New York, 2006.
- [9] O. Ivanyshyn, R. Kress and P. Serranho, Huygens' principle and iterative methods in inverse obstacle scattering, *Advances in Computational Mathematics*, 33, 413-429, 2010.
- [10] O. Ivanyshyn and R. Kress, Inverse scattering for surface impedance from phase-less far field data, *J. Computational Physics*, 230, 3443-3452, 2011.
- [11] R.V. Khachaturov, Direct and inverse problems of determining the parameters of multilayer nanostructures from the angular spectrum of the intensity of reflected X-rays, *Computational Mathematics and Mathematical Physics*, 49, 1781-1788, 2009.
- [12] M.V. Klibanov, Thermoacoustic tomography with an arbitrary elliptic operator, *Inverse Problems*, 29, 025014, 2013.
- [13] M.V. Klibanov and P.E. Sacks, Phaseless inverse scattering and the phase problem in optics, *J. Math. Physics*, 33, 3813-3821, 1992.
- [14] M.V. Klibanov, Phaseless inverse scattering problems in three dimensions, *SIAM J. Appl. Math.*, 74, 392-410, 2014.
- [15] M.V. Klibanov, On the first solution of a long standing problem: Uniqueness of the phaseless quantum inverse scattering problem in 3-d, *Applied Mathematics Letters*, 37, 82-85, 2014.
- [16] M.V. Klibanov, Uniqueness of two phaseless non-overdetermined inverse acoustics problems in 3-d, *Applicable Analysis*, 93, 1135-1149, 2014.
- [17] M.V. Klibanov and V.G. Romanov, Reconstruction formula for a 3-d phaseless inverse scattering problem for the Schrödinger equation, *J. Inverse and Ill-Posed Problems*, accepted for publication; preprint is available at www.arxiv.org, *arxiv* 1412.8201v1 [math-ph], December 28, 2014.
- [18] M.V. Klibanov and V.G. Romanov, Explicit formula for the solution of the phaseless inverse scattering problem of imaging of nano structures, *J. Inverse and Ill-Posed Problems*, 23, 187-193, 2015.
- [19] O.A. Ladyzhenskaya, *Boundary Value Problems of Mathematical Physics*, Springer, New York, 1985.
- [20] M.M. Lavrentiev, V.G. Romanov and V.G. Vasiliev, *Multidimensional Inverse Problems for Differential Equations*, Springer-Verlag, Berlin, 1970.
- [21] F. Natterer, *The Mathematics of Computerized Tomography*, John Wiley & Sons, Chichester, 1986.

- [22] Z.T. Nazarchuk, R.O. Hryniv and A.T. Synyavsky, Reconstruction of the impedance Schrödinger equation from the modulus of the reflection coefficients, *Wave Motion*, 49, 719-736, 2012.
- [23] R.G. Newton, *Inverse Schrödinger Scattering in Three Dimensions*, Springer, New York, 1989.
- [24] R.G. Novikov, A multidimensional inverse spectral problem for the equation $-\Delta\psi + (v(x) - Eu(x))\psi = 0$, *Funct. Anal. Appl.*, 22, 263–272, 1988.
- [25] R.G. Novikov, The inverse scattering problem on a fixed energy level for the two-dimensional Schrödinger operator, *J. Functional Analysis*, 103, 409-463, 1992.
- [26] R.G. Novikov, Explicit formulas and global uniqueness for phaseless inverse scattering in multidimensions, *arxiv*: 1412.5006v1, December 16, 2014, *J. Geometrical Analysis*, DOI: 10.1007/5.12220-014-9553-7, 2015.
- [27] R.G. Novikov, Formulas for phase recovering from phaseless scattering data at fixed frequency, *arxiv*: 1502.02228v2, 2015.
- [28] T. C. Petersena, V.J. Keastb and D. M. Paganinc, Quantitative TEM-based phase retrieval of MgO nano-cubes using the transport of intensity equation, *Ultramicroscopy*, 108, 805-815, 2008.
- [29] R. Phillips and R. Milo, A feeling for numbers in biology, *Proc. Natl. Acad. Sci. USA*, 106, 21465-71, 2009.
- [30] <http://kirschner.med.harvard.edu/files/bionumbers/fundamentalBioNumbersHandout.pdf>
- [31] V.G. Romanov, *Integral Geometry and Inverse Problems for Hyperbolic Equations*, Springer - Verlag, Berlin, 1974.
- [32] V.G. Romanov, *Inverse Problems of Mathematical Physics*, VNU Science Press, Utrecht, 1987.
- [33] V.G. Romanov, *Investigation Methods for Inverse Problems*, VSP, Utrecht, 2002.
- [34] V.G. Romanov, Inverse problems for differential equations with memory, *Eurasian J. of Mathematical and Computer Applications*, 2, issue 4, 51-80, 2014.
- [35] A. Ruhlandt, M. Krenkel, M. Bartels, and T. Salditt, Three-dimensional phase retrieval in propagation-based phase-contrast imaging, *Physical Review A*, 89, 033847, 2014.
- [36] B.R. Vainberg, Principles of radiation, limiting absorption and limiting amplitude in the general theory of partial differential equations, *Russian Math. Surveys*, 21, 115-193, 1966.
- [37] B.R. Vainberg, *Asymptotic Methods in Equations of Mathematical Physics*, Gordon and Breach Science Publishers, New York, 1989.